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INSTABILITY OF PLASTIC STRAIN AND FRACTURE. STRAIN DIAGRAM FOR INHOMOGENEOUS MEDIA

A. M. Avdeenko

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A strain diagram for a locally inhomogeneous medium (a medium with a porous structure or a system with noncut particles) is constructed on the basis of the Cosserat nonlinear-pseudocontinuum model. A modified criterion of geometrical softening that allows one to establish the dependence of the moment of stability loss on the statistical characteristics of the medium is considered.

The strain field that occurs under loading of a medium with pores or second-phase particles is inhomogeneous: the particles (pores) are stress concentrators and interact with one another at large values of average strains (the flow of the neighborhood of a pore near the neighboring pore is more intense). The increase in local deviations changes the averaged strain diagram for a "pure" (without pores or particles) medium, and the measure of influence is determined not only by the pore concentration but also by the statistics, i.e., by the mutual arrangement of pores (particles) or, in other words, by second or, possibly, higher-order correlation functions.

The process of fracture should be considered together with the process of deformation. First, plastic deformation initiates microcracking (the formation of pores) on structural heterogeneities, namely, on second-phase particles and decelerated shear strips and at the boundaries of grains softened by segregations. Second, the coalescence of microcracks into a mesocrack, which leads to a "hollow" viscous buckling on scales of 0.1–10.0 μm, depends on the magnitude of deformation. Third and finally, when the possibility of the plastic relaxation of external loads is exhausted as a whole, this results in the formation of a main crack, i.e., in macrofracture [1].

To construct an adequate model of fracture, it is primarily necessary to construct the strain diagram for a porous medium (the medium with second-phase particles) with allowance for the fluctuations of plastic deformation in it, relate the generalized parameters of this diagram (e.g., the effective hardening index) with heterogeneity statistics, and formulate a modified criterion of geometrical softening, which, in particular, makes it possible to solve the structural-optimization problem, i.e., to determine a relation between the number and distribution of second-phase particles (pores) for a given diagram of a "pure" medium that ensures the maximum macrouniform deformation. To tackle these questions, it is necessary to modify the statistical nonlinear-pseudocontinuum model for making allowance for local structural heterogeneities, which was considered in [2, 3].

For a statistical description of slow (scleronomous) deformation, we introduce the density-distribution functional of displacement-field fluctuations $A_\mu(\boldsymbol{r})$ ($\mu = 1,\ 2,\ 3$): $f[A_\mu] = \exp\left(-W[A_\mu]\right)$. We present the generating functional of the system considered in the form of the functional series

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$$W[A_\mu] = \int \ldots \int \sum_{k=2}^{\infty} \frac{V_k^{\mu p \ldots q\nu}(\boldsymbol{r}_i)}{k} A_{\mu,p} \ldots A_{q,\nu}\, d\boldsymbol{r}_1 \ldots d\boldsymbol{r}_i \ldots . \tag{1}$$

The real tensors $V_k^{\mu\ldots\nu}(\boldsymbol{r}_i)$ of rank $2k$ are called the vertices, the first k indices ($\mu = 1, 2, 3$) are referred to the components of the displacement field A_μ, and the subsequent k indices ($\mu = 1, 2, 3$) are referred to the spatial derivatives $A_{\mu,p} = \partial A_\mu / \partial x_p$ and $\boldsymbol{r} = (x_1, x_2, x_3)$.

The change in the field $A_\mu(\boldsymbol{r})$ with the deformation time t is called the loading trajectory $A_\mu(\boldsymbol{r}, t)$. The distribution density $f[A_\mu]$ is a monotone function $W[A_\mu]$; therefore, the trajectory $\bar{A}_\mu$ satisfying the variational equation $\delta W[A_\mu]/\delta A_\mu = 0$ under given boundary conditions corresponds to the most probable process. Its solution $\bar{A}_\mu$ is called the "classical" trajectory, and the difference $\delta A_{\mu,\nu} = A_{\mu,\nu} - \bar{A}_{\mu,\nu}$ is called the fluctuations. Hereinafter, we confine ourselves only to the so-called "active" trajectories. The length of the trajectory $s = \int\limits_{t_0}^{t} \Big(\dfrac{\partial \bar{A}_{\mu,\nu}}{\partial t} \dfrac{\partial \bar{A}^{\mu,\nu}}{\partial t}\Big)^{1/2} d\tau$, where $\bar{A}_{\mu,\nu} = \partial \bar{A}_\mu(\boldsymbol{r}, t)/\partial x_\nu$, increases during loading, and the "active" evolution along the "classical" trajectory is the same for all microvolumes v_i (M-sample in the concepts of Il'yushin's school [4]).

To construct the generating functional of strain-field fluctuations, we expand the functional (1) into a series in the vicinity of the "classical" trajectory $\bar{A}_{\mu,\nu}$. If the vertices in (1) have the maximum order n, the vertex of fluctuations $\bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i)$ has the form

$$\bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i) = \bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i) + \int \ldots \int \sum_{p=3}^{n} V_{p,r\ldots q}^{\mu\ldots\nu}(\boldsymbol{r}_i, \boldsymbol{r}_i') \bar{A}^{r,l}(\boldsymbol{r}_1', t) \ldots \bar{A}^{s,q}(\boldsymbol{r}_p', t)\, d\boldsymbol{r}_1' \ldots d\boldsymbol{r}_p'.$$

Integration over $\boldsymbol{r}_1' \ldots \boldsymbol{r}_p'$ on the "M-sample" gives a negligible constant proportional to the volume of solid to the power $p - k$. We put the vector in the m-dimensional space E_m (m is the number of independent components of the tensor $\bar{A}^{r,l}$) into correspondence with the tensor $\bar{A}^{r,l}$ and represent the combination $\bar{A}^{\mu,\nu} \ldots \bar{A}^{p,s}$ in the form of the internal-geometry function of a "classical" trajectory, i.e., its length s, curvatures $\vartheta_1(s) \ldots \vartheta_{n-1}(s)$, and torsion $\vartheta_n(s)$ [4]. In this case, we have $\bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i, t) = \bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i, \vartheta_n(s), s)$. Hereinafter, we restrict ourselves to the consideration of simple processes (proportional loading) for which the scalar curvatures and torsions are identically equal to zero. Then, $\bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i) = \bar{V}_k^{\mu\ldots\nu}(\boldsymbol{r}_i, s)$, i.e., the generating functional for simple (proportional) loading of the "M-sample" is parametrized by the second invariant of the tensors of displacement-field derivatives. Hereinafter, the bar above the vertex and δ before fluctuations are omitted.

The normalized Gaussian mean with weight $\exp(-W)$ for $V_k^{\mu\ldots\nu} = 0$ ($k > 2$) is called the free correlation function of deformation, and we present it in the form

$$R_{20}^{\mu\ldots\nu}(\boldsymbol{r}) = C_2^{\mu\ldots\nu} R_{20}(\boldsymbol{r}) = \langle A^{\mu,p}(\boldsymbol{r}') A^{q,\nu}(\boldsymbol{r}' + \boldsymbol{r}) \rangle$$

$$= \int A^{\mu,p}(\boldsymbol{r}') A^{q,\nu}(\boldsymbol{r}' + \boldsymbol{r}) \exp(-W[A_\mu])\, dA_\mu \Big/ \int \exp(-W[A_\mu])\, dA_\mu. \tag{2}$$

Here dA_μ is the symbol of continual integration and $C_2^{\mu\ldots\nu}$ is a certain symmetric tensor.

The operator $V_{20}^{\mu\ldots\nu}(\boldsymbol{r}_i)$, which is inverse to the free correlation function $R_{20}^{\mu\ldots\nu}(\boldsymbol{r})$, is determined by means of the relation

$$\int V_2^{\mu\ldots\nu}(\boldsymbol{r}_1, \boldsymbol{r}_1') R_{20,mpqn}(\boldsymbol{r}_1' - \boldsymbol{r}_2)\, d\boldsymbol{r}_1' = \delta_m^\mu \delta_n^\nu \delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) \tag{3}$$

and is called the second-degree free vertex. For a system with $V_k^{\mu\ldots\nu}(\boldsymbol{r}_i) = 0$ ($k > 2$), the second-order free vertex coincides with the vertex $V_2^{\mu\ldots\nu}(\boldsymbol{r}_i)$. Generally, when $V_k^{\mu\ldots\nu}(\boldsymbol{r}_i) \neq 0$ ($k > 2$), the normalized two-point mean with weight $\exp(-W)$ determines the full correlation function $R_2^{\mu\ldots\nu}(\boldsymbol{r})$. The operator $V_2^{\mu\ldots\nu}(\boldsymbol{r}_i)$, which is inverse to the full correlation function, is given by expression (3) with the substitution $R_{20}^{\mu\ldots\nu}(\boldsymbol{r}) \to R_2^{\mu\ldots\nu}(\boldsymbol{r})$. This operator takes into account the interactions between the strain-field fluctuations (nonlinear effects) and, hereinafter, this operator is called the second-degree full vertex. Generally speaking, the full vertex does not coincide with the operator for the squared field variables in the generating functional (1); now it is denoted by $V_{20}^{\mu\ldots\nu}(\boldsymbol{r}_i)$.

We assume that in the initial (unloaded) state, the "classical" trajectory corresponds to the equation of equilibrium of the Cosserat elastic pseudocontinuum model [5]

$$\nabla^2 A_\mu + \frac{1}{1-2\nu}\nabla_\mu(\nabla_\nu A^\nu) - \xi_0^2\nabla^2(\nabla^2 A_\mu - \nabla_\mu\nabla^\nu A_\nu) = 0,$$

where ξ_0 is the structural scale of the elastic pseudocontinuum, $\nabla_\mu = \dfrac{\partial}{\partial x_\mu}$, and $\nabla^2 = \displaystyle\sum_{i=1}^{3}\frac{\partial^2}{\partial x_\mu^2}$.

We divide the field A_μ into longitudinal and transverse components: $A_\mu = A_\mu^n + A_\mu^t$; then the corresponding distortions are $A_{\mu,\nu}^n = 1/(n\delta_{\mu\nu}A_{k,k})$ and $A_{\mu,\nu}^t = A_{\mu,\nu} - A_{\mu,\nu}^n$. We present the second-degree free vertex in the form of the sum $V_{20}^{\mu\ldots\nu}(\boldsymbol{r}_i) = V_{20}^{\mu\ldots\nu,n}(\boldsymbol{r}_i) + V_{20}^{\mu\ldots\nu,t}(\boldsymbol{r}_i)$:

$$V_{20}^{\mu\ldots\nu,n}(\boldsymbol{r}_i, s\to+0) = \frac{T_2^{\mu\ldots\nu,n}}{V\langle\varepsilon_2^2\rangle}\Big[\frac{3-2\nu}{1-2\nu}\Big]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

$$V_{20}^{\mu\ldots\nu,t}(\boldsymbol{r}_i, s\to+0) = \frac{T_2^{\mu\ldots\nu,t}}{V\langle\varepsilon_1^2\rangle}[1+\xi_0^2\nabla^2]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

where $\langle\varepsilon_1^2\rangle = V^{-1}\displaystyle\int R_{20\mu\nu}^{\mu\nu,t}(\boldsymbol{r})\,d\boldsymbol{r}$ and $\langle\varepsilon_2^2\rangle = V^{1}\displaystyle\int R_{20\mu\nu}^{\mu\nu,n}(\boldsymbol{r})\,d\boldsymbol{r}$ are the transverse and longitudinal variances of the strain-field fluctuations in the unloaded state (V is the volume of solid).

One can show that in the loaded state, the free vertices of longitudinal fluctuations remain unchanged, and those of transverse fluctuations take the form

$$V_{20}^{\mu\ldots\nu,t}(\boldsymbol{r}_i, s\to+0) = \frac{T_2^{\mu\ldots\nu,t}}{V\langle\varepsilon_1^2\rangle}[\theta(s)+\xi_0^2\nabla^2]\delta(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

where $\theta(s) = (1/G)\,d\tau/ds$ is the tangent hardening modulus along the "classical" trajectory normalized to the modulus of shear.

The corresponding free correlation functions have the form

$$R_{20}^{\mu\ldots\nu,t}(\boldsymbol{r},s) = \frac{C_2^{\mu\ldots\nu,t}V\langle\varepsilon_1^2\rangle}{4\pi r\xi_0^2}\exp(r/\xi), \qquad R_{20}^{\mu\ldots\nu,n}(\boldsymbol{r},s) = C_2^{\mu\ldots\nu,n}V\langle\varepsilon_{12}^2\rangle\delta(\boldsymbol{r}_1-\boldsymbol{r}_2),$$

where $C_2^{\mu\ldots\nu,t} = \delta^{\mu p}\boldsymbol{e}^q\boldsymbol{e}^\nu$ and $C_2^{\mu\ldots\nu,n} = \boldsymbol{e}^\mu\boldsymbol{e}^\mu\boldsymbol{e}^\nu\boldsymbol{e}^\nu$ ($\boldsymbol{e}^\nu$ is the unit vector in the direction $\boldsymbol{r}$), $\xi = \xi_0\theta^{-\alpha/2}$ is the correlation interval for the transverse-strain fluctuations [$\alpha = 1-(n+2)g_4/2$, where the quantity g_4 is related to the fourth-degree vertex of strain fluctuations by the relation $\displaystyle\int V_4^{\mu\ldots\nu}(\boldsymbol{r}_i, s\to+0)\,d\boldsymbol{r}_i = T_4^{\mu\ldots\nu}g_4$], and n is the number of components of the field A_μ [2, 3].

Accordingly, the variances of the fluctuations of transverse and longitudinal-strain fields are expressed as follows:

$$\langle\varepsilon^2(\theta)\rangle = V^{-1}\int R_{20\mu\nu}^{\mu\nu,t}(\boldsymbol{r},\theta)\,d\boldsymbol{r} = \langle\varepsilon_1^2\rangle\theta^{-\alpha}, \qquad \langle\varepsilon^2(\theta)\rangle = V^{-1}\int R_{20\mu\nu}^{\mu\nu,n}(\boldsymbol{r},\theta)\,d\boldsymbol{r} = \langle\varepsilon_2^2\rangle;$$

because $\theta \ll 1$ and $\langle\varepsilon_1^2\rangle \approx \langle\varepsilon_2^2\rangle$, except for the narrow region in the vicinity of macroelasticity, we have $\langle\varepsilon^2(\theta)\rangle \gg \langle\varepsilon_2^2\rangle$. Hereinafter, we consider only the statistics of transverse strain-field fluctuations.

The dimensionless hardening modulus decreases during loading ($d\theta/ds < 0$), which leads to the power-type singularities $\xi(\theta)$ and $\langle\varepsilon^2(\theta)\rangle$ with preservation of the similarity $\langle\varepsilon_1^2\rangle/\langle\varepsilon^2(\theta)\rangle = \xi_0^2/\xi^2(\theta)$. For the experimental dependences $\xi(\theta)$ and $\langle\varepsilon^2(\theta)\rangle$ for iron [6] and aluminum [2] polycrystals, the value of the index α lies in the interval 1.1–1.2. Extrapolation of $\xi(\theta)$ to the unloaded state $s\to+0$ gives the structural scale $\xi_0 = 25$–100 μm.

We pass to the reduced variables: $A_{\mu,\nu} \to A_{\mu,\nu}\xi_0 V^{-1/2}\langle\varepsilon_1^2\rangle^{-1/2}$; then, in the state $s > 0$, the second-degree free vertex is $V_{20}^{\mu\ldots\nu}(\boldsymbol{r}_i) = T_2^{\mu\ldots\nu}(\theta(s)\mu^2 + \nabla_\mu\nabla^\mu)\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$. After the Fourier transform

$$V_{20}^{\mu\ldots\nu}(\boldsymbol{p}) = \int V_{20}^{\mu\ldots\nu}(\boldsymbol{r},\boldsymbol{r}_1)\exp(i(\boldsymbol{pr} + \boldsymbol{p}_1\boldsymbol{r}_1))\,d\boldsymbol{r}\,d\boldsymbol{r}_1,$$

we have $V_{20}^{\mu\ldots\nu}(p) = T_2^{\mu\ldots\nu}(\theta(s)\mu^2 + p^2)$, where $\mu = \xi_0^{-1}$. As a result, the dimensionless hardening modulus $\theta(s)$ in the state $s > 0$ is related to $V_{20}^{\mu\ldots\nu}(\boldsymbol{p},\theta)$ by the relation

$$\theta(s) = \lim_{\boldsymbol{p}\to 0}(V_{20}(\boldsymbol{p},\theta)\mu^{-2}). \tag{4}$$

The disperse heterogeneities are taken into account under the assumption that the quantity λ^2 depends on the spatial coordinate: $\lambda^2 = \lambda^2(\boldsymbol{r})$. We determine the mean

$$\bar{\lambda}^2 = V^{-1}\int \lambda^2(\boldsymbol{r})\,d\boldsymbol{r}$$

and introduce the random function $\varphi(\boldsymbol{r}) = \mu^{-2}(\lambda^2(\boldsymbol{r}) - \bar{\lambda}^2)$.

In the initial state $s \to +0$, for media with a given structural scale, the mean $\bar{\lambda}^2$ $(s \to +0)$ tends to $\lambda^2 \to \mu^2(1+\eta N_0)$, where N_0 is the volume portion of heterogeneities. The quantity $\eta = (G - G_1)/G$ (G_1 and G are the elastic moduli of the discrete heterogeneity and the medium, respectively) determines the type of heterogeneity: if the structural heterogeneity is a pore, then $\eta = -1$. In the general case ($\eta \geqslant -1$), a particle with a larger elastic modulus corresponds to a positive value of η. It is assumed that the heterogeneities do not form a coupled cluster, and their average size is much smaller than the structural scale of the medium, and the random function $\varphi(\boldsymbol{r})$ implements a delta-correlated (the pores or particles do not overlap one another) isotropic process: $\langle\langle\varphi(\boldsymbol{r}_1)\varphi(\boldsymbol{r}_2)\rangle\rangle = \Delta(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \Delta\delta(\boldsymbol{r}_1 - \boldsymbol{r}_2)$, where the angular brackets mean averaging over all realizations of the random field $\varphi(\boldsymbol{r})$. For pores and particles, we have $\Delta = N_0(1 - N_0)$ and $\Delta = \eta^2 N_0(1 - N_0)$, respectively.

The generating functional for the field φ has the form

$$W_1[\varphi] = \int \frac{\varphi^2}{2\Delta}\,dr,$$

and the common generating functional for a system with local heterogeneities is $W'[A_\mu,\varphi] = W[A_\mu] + \Delta W[A_\mu,\varphi] + W_1[\varphi]$, where $W[A_\mu]$ is the generating functional in the nonlinear pseudocontinuum model

$$\Delta W[A_\mu,\varphi] = \int \frac{\mu^2}{2}\,T_2^{\mu pq\nu}\varphi A_{\mu,p}A_{q,\nu}\,d\boldsymbol{r}.$$

Averaging the system with the generating functional $W'[A_\mu,\varphi]$ over the field φ in the continual meaning, we obtain the generating functional of strain-field fluctuations in the nonlinear-pseudocontinuum model with local heterogeneities

$$\exp(-W''[A_\mu]) = \int \exp(-W[A_\mu,\varphi])\,d\varphi$$

or

$$W'' = -\ln\int \exp(-W[A_\mu,\varphi])\,d\varphi,$$

where $d\varphi$ is the symbol of continual integration over the field φ.

The normalized two-point means with weight $\exp(-W''[A_\mu])$ are the full correlation functions $R_2^{\mu\ldots\nu}(\boldsymbol{r})$ in the nonlinear-pseudocontinuum model with heterogeneities. The operator $V_2^{\mu\ldots\nu}(\boldsymbol{r}_i)$ (the second-degree full vertex), which is inverse to $R_2^{\mu\ldots\nu}(\boldsymbol{r})$, now depends on the heterogeneity statistics, namely, on the quantity Δ. By analogy with (4), the corresponding dimensionless hardening modulus is determined by the relation $\Omega(s) = \lim_{\boldsymbol{p}\to 0} V_2(\boldsymbol{p},\Delta,\theta(s))$. The effective stress along the "classical" trajectory has the form

$$\sigma(s) = \int_1^{\theta(s)} \Omega(\theta)\,\frac{ds(\theta)}{d\theta}\,d\theta. \tag{5}$$

Thus, the construction of strain diagrams for nonlinear media with a specified statistics of uncorrelated local heterogeneities is reduced to averaging of the functional $W'[A_\mu, \varphi]$ over the field φ and to the calculation of the second-degree full vertex $V_2^{\mu\ldots\nu}(\boldsymbol{p})$ at the point $\boldsymbol{p} = 0$ with subsequent integration of (5).

Let $V_k^{\mu\ldots\nu}(\boldsymbol{r}_i) = 0$ $(k > 2)$ in the initial model; then the single nonlinearity in the system with local heterogeneities is related to the term $\varphi A_{\mu,\nu} A^{\mu,\nu}$. The variance of the local heterogeneity $\Delta \approx N_0 \ll 1$ is regarded as a small expansion parameter; then the paired correlation function in the model with heterogeneities in the state $s > 0$ has the form $R_2^{\mu\ldots\nu}(\boldsymbol{p}) = R_{20}^{\mu\ldots\nu}(\boldsymbol{p}) - R_{20}^{\mu\ldots q'}(\boldsymbol{p})\Sigma_{\mu'\ldots\nu'}(\boldsymbol{p})R_{20}^{p\ldots\nu'}(\boldsymbol{p})$, where $R_{20}^{\mu\ldots\nu}(\boldsymbol{p}) = C_2^{\mu\ldots\nu}(\mu^2\theta(s) + p^2)^{-1}$ and $\Sigma_{\mu'\ldots\nu'}(\boldsymbol{p}) = \Delta T_{2,\mu'\ldots\nu'}\Big(\int R_{20}(\boldsymbol{q})\,d\boldsymbol{q} + 2\Delta \int R_{20}^2(\boldsymbol{q})\,d\boldsymbol{q} + \ldots\Big) + O(\Delta^3)$.

Taking into account that $C_2^{\mu\nu pq} T_{2,k\nu pq} = \delta_k^\mu$ and $V_2(\boldsymbol{p}) = R_2^{-1}(\boldsymbol{p})$ and solving this expression relative to $V_2(\boldsymbol{p})$, we have

$$V_2(\boldsymbol{p}) = V_{20}(\boldsymbol{p}) + \Sigma(\boldsymbol{p}, \Delta) = V_{20}(\boldsymbol{p}) - \bar{\Delta}(\theta) \int R_{20}(\boldsymbol{q})\,d\boldsymbol{q},$$

where $\bar{\Delta}(\theta) = \Delta + 2\Delta^2 \int R_{20}^2(\boldsymbol{q})\,d\boldsymbol{q} + O(\Delta^3)$.

Passing to the limit $\boldsymbol{p} \to 0$ and taking into account the definition (5), we obtain an expression for the dimensionless hardening modulus of a locally inhomogeneous medium in the form

$$\Omega(\theta) = (1 + \eta N_0)(\theta + \mu^{-2}\bar{\Delta}(\theta)) \int R_{20}(\boldsymbol{q})\,d\boldsymbol{q}, \qquad \bar{\Delta}(\theta) = \Delta + 2\Delta^2 \int R_{20}^2(\boldsymbol{q})\,d\boldsymbol{q} + \ldots\ .$$

In calculating the integrals $J_1 = \int R_{20}(\boldsymbol{q})\,d\boldsymbol{q}$ and $J_2 = \int R_{20}^2(\boldsymbol{q})\,d\boldsymbol{q}$, to exclude singularities, a regularization determined in such a way that $\Omega(\theta) = \theta(s \to +0)(1 + \eta N_0) = 1 + \eta N_0$ in the unloaded state is needed [3, 6]. The corresponding regularized integrals have the form

$$J_1(\text{reg}) = \int R_{20}(\boldsymbol{q})\,d\boldsymbol{q} = -\frac{1}{2}\,\mu^2\theta \ln\,\theta, \qquad J_2(\text{reg}) = \int R_{20}^2(\boldsymbol{q})\,d\boldsymbol{q} = -\frac{1}{2}\ln\theta.$$

Finally, we obtain

$$\Omega(\theta) = (1 + \eta N_0)\theta\Big(1 + \frac{1}{2}\,\bar{\Delta}(\theta)\ln\theta + \ldots\Big), \qquad \bar{\Delta}(\theta) = \Delta(1 - \Delta \ln\,\theta + \ldots). \tag{6}$$

It suffices to use relations (6) for construction of the strain diagram for a nonlinear medium with heterogeneities: the first relation involves local overloading of the structure $[\Omega(\theta) \approx \Delta \ln\,\theta]$, and the second takes into account the effective interaction between local heterogeneities $[\bar{\Delta}(\theta) \approx \Delta^2 \ln\,\theta]$. It is important that the relations obtained for Ω and $\bar{\Delta}$ do not include the structural-scale quantity ξ_0.

One can refine relation (6) in the following manner. We introduce the functions $F_1(\theta) = \Omega/\theta$ and $F_2(\theta) = \bar{\Delta}(\theta)/\Delta$. Since $F_1(1) = F_2(1) = 1$, the equalities $F_1(\theta_1)/F_1(\theta_2) = 1/F_1(\theta_1/\theta_2)$ and $F_2(\theta_1)/F_2(\theta_2) = 1/F_2(\theta_1/\theta_2)$ hold, which is equivalent to the system of renormalization-group equations

$$\theta\,\frac{d\ln\,\Omega}{d\theta} = \theta\,\frac{d\ln\Omega}{d\theta}\Big|_{\theta=1} + \frac{d\ln\Omega}{d\Delta}\,\frac{d\Delta}{d\theta}\Big|_{\theta=1}, \qquad \theta\,\frac{d\bar{\Delta}}{d\theta} = \theta\,\frac{d\ln\Delta}{d\theta}\Big|_{\theta=1}.$$

For the initial condition $\Omega(s \to +0) = 1 + \eta N_0$, the solution of this system has the form

$$\Omega(\theta) = (1 + \eta N_0)\theta \exp\Big(\int_\Delta^{\bar{\Delta}} \frac{A(\Delta)}{B(\Delta)}\,d\Delta\Big),$$

where $A(\Delta) = \dfrac{d\ln\Omega}{d\ln\theta}\Big|_{\theta=1}$ and $B(\Delta) = \dfrac{d\bar{\Delta}}{d\theta}\Big|_{\theta=1}$.

The first terms of the expansion of $\Omega(\theta)$ in power of Δ coincide with expression (4); however, the approximation is more correct owing to the effective summation of the terms containing no matter how high a power of Δ.

Calculating in the first approximation of Δ, we have

$$\Omega(\theta) = (1 + \eta N_0)\theta^{1+\nu(\theta)}, \tag{7}$$

where $\nu(\theta) = (1/2)\ln(1 + 2\Delta \ln\theta)/\ln\theta = (1/2)\bar{\Delta}(\theta) \approx \Delta(1 - \Delta\ln\theta)^{-1} + O(\Delta^3)$.

The substitution of (7) into (5) allows to one to obtain a strain diagram for a "pure" medium ($N_0 = 0$) that takes into account the heterogeneity statistics.

The explicit integration of (5) is possible only in the first order over Δ for some particular dependences $\theta(s)$. Let the "classical" trajectory in real coordinates be approximated by the dependence $\sigma(s) = \sigma_0 s^m$ ($m < 1$).

In the unloaded state ($s \to +0$), we have $\theta(s) \approx s^{m-1} \to \infty$; therefore, we confine ourselves to the consideration of the strain $s \geqslant s_1$ and note that $\lim\limits_{s\to s_1} \theta(s) \to 1 - 0$. Hence, the corresponding stress value is $\sigma(s_1) = \sigma_1 = m^{-1}(\sigma_0 m)^{1/(1-m)}$. Substituting this dependence into (3), after integration we obtain $\sigma(\theta) = (1+\eta N_0)\sigma_1(1 + (m/M)(\theta^{M/(m-1)} - 1)) = (1+\eta N_0)\sigma_1(1+(m/M)((s/s_1)^M - 1))$, where $M = m - \Delta(1-m)$.

For the majority of plastic materials $\sigma_0 = (2\text{–}9)\cdot 10^{-3}$ (the real stress is normalized by the modulus of shear), $m = 0.2$–0.3 [1] and, hence, $s_1 = 10^{-4}$–10^{-3} and $s/s_1 \gg 1$ for actual processes; therefore, $\sigma(s) \approx \sigma_0 s^M$. The hardening index M is always smaller than that for a "pure" medium; we note that the relative error $(M - m)/m = \Delta(1-m)/m \approx \eta^2 N_0(1-m)/m$ increases with increase in N_0 and η and with a decrease in m. In the range of small $N_0 < 0.02$, the quantity $(M - m)/m$ is comparable with the variance of reproducibility of the hardening index of the "pure" medium $\delta m/m$, and its contribution can be ignored [7]. A decrease in the dimensionless hardening modulus θ leads to an increase in Δ, i.e., to a decrease in M irrespective of the sign of η (for $-\Delta\ln\theta \approx 1$, it is necessary to take into account the highest orders of the perturbation theory).

For active simple loading, we define the effective hardening index

$$M(\theta) = \frac{d\ln\sigma(\theta)}{d\ln\theta}\,\frac{d\ln\theta}{d\ln s} = \frac{\Omega(\theta)s(\theta)}{\sigma(\theta)},$$

where $s(\theta)$ is the real strain and $\Omega(\theta)$ and $\sigma(\theta)$ are the effective hardening modulus and the stress, respectively.

In the unloaded state, $s \to +0$ ($\theta \to 1-0$); therefore, $\sigma(\theta \to 1-0) = \lim\limits_{s\to+0} \theta(s)s(\theta)$ and $M(s \to +0) = 1$ (elastic system).

We now consider a simple active process, namely, uniaxial tension. Let Σ be the current cross section of the sample and F be the applied force. The plastic flow is stable for $dF = -\sigma d\Sigma + \Sigma d\sigma > 0$ or $d\ln\sigma/d\ln s > s$, because $d\Sigma = -\Sigma\, ds$. Otherwise, $d\ln\sigma/d\ln s < s$ and the macrohomogeneous flow is unstable: upon uniaxial tension, a neck forms ($d\ln\sigma/d\ln s = s$). For the power approximation of the true diagram $\sigma = \sigma_0 s^m$, the solution of the equation $d\ln\sigma/d\ln s = s$ has a simpler form, $s_{\text{uni}} = m$. The flow is stable for $s > m$ and the localization occurs for $s < m$.

Within the framework of the concept considered, taking into account fluctuation corrections requires a replacement of the index m by the effective quantity $M(\theta)$. The modified criterion of the loss of flow stability takes the form

$$M(\theta) - s(\theta) = 0 \tag{8}$$

or

$$\sigma(\theta) - \Omega(\theta) = 0. \tag{9}$$

The unique solution of Eqs. (8) and (9) relative to θ (the quantity θ_{uni}) and the corresponding uniform strain s_{uni} depend on the variance of Δ and the strain-diagram parameters, because $\theta(s)$ is a smooth decreasing function of real strain.

In the zero approximation of the perturbation theory, we have $s^0_{\text{uni}} = m$, and Eqs. (8) and (9) should be solved numerically in highest-order approximations. For a pure iron-based porous structure ($\sigma_0 = 8.63\cdot 10^{-3}$, $m = 0.31$, and $\eta = -1$), the following results were obtained: an increase in the pore concentration reduces the uniform strain from $s_{\text{uni}} \approx 0.27$ for $N_0 = 0.05$ to $s_{\text{uni}} \approx 0.14$ for $N_0 = 0.15$. If $\Delta \ll m/(1-m)$, one

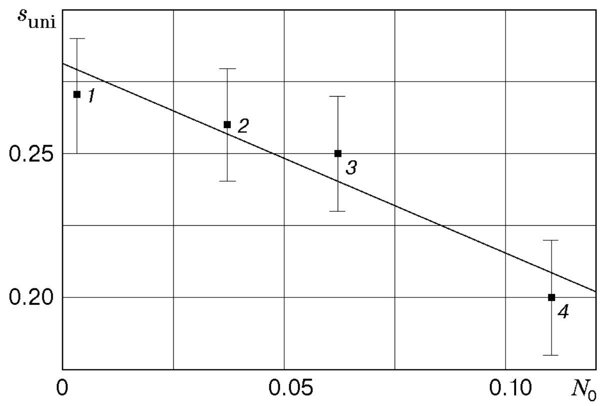

Fig. 1. Uniform strain of porous iron for $N_0 =$ 0.003 (1), 0.037 (2), 0.062 (3), and 0.110 (4).

can ignore the change in Δ during loading: $\Delta \ll \ln^{-1}\theta^0_{\text{uni}}$, where $\theta^0_{\text{uni}} = \theta_{\text{uni}}|_{s=m} \ll 1$. Then, $s_{\text{uni}}(\Delta) = s_{\text{uni}}(0) - \Delta(1-m)$, where $s_{\text{uni}}(0) = m$ is the macrouniform strain of the "pure" medium. For a porous structure, $\Delta = N_0(1-N_0)$ and, hence, for $N_0 \ll 1$, we have

$$s_{\text{uni}}(N_0) = m - N_0(1-m). \tag{10}$$

We compare this relation with the experiment for porous iron, which was obtained by sintering and compaction with a different volume portion of pores N_0 [7]. The technological pores are distributed isotropically and uniformly. Their diameter, 1.5–3.0 μm, is much smaller than the structural scale $\xi_0 = 20$–30 μm of the elastic pseudocontinuum determined for a matrix from strain-profile statistics [6]. Figure 1 shows the macrouniform strain (uniaxial tension) versus the pore concentration N_0 according to the data of [7] with allowance for the standard estimate of the r.m.s. deviation. A decrease in the pore concentration leads to a decrease in the macrouniform strain from approximately 0.27 for $N_0 = 0.003$ to approximately 0.2 for $N_0 = 0.1$. For $N_0 < 0.11$, the dependence $s_{\text{uni}}(N_0)$ is linear: $s_{\text{uni}}(N_0) = (0.279 \pm 0.009) - (0.659 \pm 0.145)N_0$. The free term 0.279 ± 0.009 and the angle of slope 0.659 ± 0.145 coincide, within the reproducibility error, with the hardening index of the base (pure iron) $m = 0.27 \pm 0.02$ and the quantity $1 - m = 0.63 \pm 0.02$, respectively, which agrees with (10) and supports the validity of the scheme proposed for construction of the strain diagram for an inhomogeneous medium.

Thus, when the average distance between the isotropic pores $N_0^{-1/3}$ is not smaller than 2–3 pore diameters, the uniform strain is well determined by means of the continuum strain diagram in the limit of the delta-correlated model with allowance for only the two-point correlation functions of strain fields, although the pore growth and microcracking are not taken into account up to values of the true strains approximately equal to 0.2.

The model considered is only the first approximation of the real structure of a locally inhomogeneous medium. The effects of the following order are connected, obviously, with the replacement of second-phase particles (pores) by clusters, when the correlation interval (half-cycle of the structure) l_c can be compared with the magnitude of the structural scale ξ_0. It calls for making allowance for corrections of order l_c/ξ_0 to the magnitude Δ.

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